

# Correlation formulas for Markovian network processes in a random environment

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March 3, 2015

## Abstract

We consider Markov processes, which describe e.g. queueing network processes, in a random environment which influences the network by determining random breakdown of nodes, and the necessity of repair thereafter. Starting from an explicit steady state distribution of product form available in the literature, we notice that this steady state distribution does not provide information about the correlation structure in time and space (over nodes). We study this correlation structure via one step correlations for the queueing-environment process. Although formulas for absolute values of these correlations are complicated, the differences of correlations of related networks are simple and have a nice structure. We therefore compare two networks in a random environment having the same invariant distribution, and focus on the time behaviour of the processes when in such a network the environment changes or the rules for traveling are perturbed. Evaluating the comparison formulas we compare spectral gaps and asymptotic variances of related processes.

## 1 Introduction

We consider stochastic networks of the Jacksonian type in a random environment. For a general introduction to Markov processes in random environments with applications to networks,

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<sup>\*</sup>Work supported by Deutscher Akademischer Austauschdienst

<sup>†</sup>Work supported by NCN Research Grant DEC-2011/01/B/ST1/01305

<sup>0</sup>*Key Words:* Product form networks, space-time correlations, spectral gap, asymptotic variance, Peskun ordering

*AMS (1991) subject classification:* 60K25, 60J25

*Short title:* correlations in networks

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see [Zhu94], [Eco05], [BM13]. These stochastic networks have recently found interest as a general model for queueing networks in connection with other areas of Operations Research, e.g. inventory theory and reliability theory. The interaction of network and environment is (i) in the queueing-inventory system that the service process decreases inventories and the inventory restricts serving customers due to limited stock at hand, and (ii) in the queueing-reliability system that external forces let servers break down which requires repair.

We concentrate in the present paper on the second framework: External forces generate random breakdowns of servers in the network and the subsequent repair is also performed under random influences. We allow the environment to be of a rather general structure, which implies that nodes may break down in isolation or in groups, and that batch repair is possible as well.

For this framework there is a product form extension of Jackson's steady state result at hand, which provides in case of ergodicity the joint steady state distribution of the environment (represented by the set of broken down nodes) and the joint queue length vector in a product form: The environment status and the queue lengths seem to decouple asymptotically and in steady state (which is the essence of Jackson's theorem in case of pure queueing systems). Clearly, this does not mean that environment and queue lengths are independent: The environment is assumed here to be a Markov process for its own, but it strongly influences the service provided by the nodes and even the arrival streams there and, furthermore, the nodes interact as well - the interactions are carried by the traveling customers. These dependencies are not expressed by the one-dimensional (in time) marginal process distribution, which is a multidimensional (in space) product form distribution. In fact, very little is known about the dependence structure of the interacting processes. Therefore we study the correlation structure in time of the environment-queue length process via the one-step correlations (the Dirichlet forms of the associated Markov process), which in time as well as in space exhibit complex dependence behaviour.

To be more precise, our interest is focused on the following scenarios: Compare two networks in a random environment which have the same invariant product form distribution, and so are in some sense variants of one another. Typical questions are: What happens to the time behaviour of a network when in such a network the rules for traveling (routing chains) are perturbed, or, when the environment changes?

Our main results are comparison theorems and formulas which provide differences of one step correlations in related, resp. perturbed networks. Although the formulas for absolute values of the one step correlations are rather complicated, it turns out that differences of correlations of related networks are surprisingly simple and have a nice structure. As a consequence, whenever we have obtained quantities connected to one step correlations for some reference network as an anchor (possibly from simulations or numerical evaluations), we can perform easily explicit performance analysis, especially sensitivity analysis by varying, e.g., breakdown and repair probabilities or routing probabilities.

The structure of the paper is as follows. In Section 2 we describe stochastic networks and the influence of the environment via Markovian breakdown and repair processes, which results in a non-Markovian structure of the queue size processes alone, and cite the steady state distribution for these networks. In Section 3 we derive the explicit formulas for the one step correlations in time for the joint environment-network process and show that for the interesting comparison problems these formulas simplify considerably. In Section 4 we show that our results allow to compare the spectral gaps and asymptotic variances of different

systems by evaluating our previous formulas suitably. Comparison results for spectral gaps allow to compare speed of convergence to stationarity for networks in  $L^2$  norm. In Section 5 we discuss relations to networks with finite buffers and extensions of our theorems to this area. Section 6 comprises the main technical proofs.

### Notation and conventions:

For a set  $M$  we denote by  $2^M = \mathcal{P}(M)$  the set of all subsets of  $M$ .

For sets  $A, B$  we write  $A \subseteq B$  for  $A$  which is a subset of  $B$  or equals  $B$ , and we write  $A \subset B$  for  $A$  which is a subset of  $B$  but does not equal  $B$ .

$\delta_{xy}$  is the Kronecker Delta, which is 1 iff  $x = y$  and 0 otherwise.

Throughout, the node set of our graphs (networks) are denoted by  $\tilde{J} := \{1, \dots, J\}$ , and the "extended node set" is  $\tilde{J}_0 := \{0, 1, \dots, J\}$ , where "0" refers to the external source and sink of the network.

We denote the diagonal matrix with a vector  $\xi$  on the diagonal and zero otherwise by  $\text{diag}(\xi)$ .  $e_j$  is the standard  $j$ -th base vector in  $\mathbb{N}^J$  if  $1 \leq j \leq J$  and  $e_0$  is the  $J$ -dimensional zero vector.  $Id$  is the identity matrix of appropriate dimension, defined by the context.

For  $D \subseteq \tilde{J}$  and  $\underline{n} = (n_j : j \in \tilde{J}) \in \mathbb{N}^J$  we write  $\underline{n}_D := (n_j : j \in D) \in \mathbb{N}^{|D|}$  and  $\underline{n}_{\tilde{J} \setminus D} := (n_j : j \in \tilde{J} \setminus D) \in \mathbb{N}^{|\tilde{J} \setminus D|}$ , and will, as usual, identify  $\underline{n} = ((n_j : j \in D), (n_j : j \in \tilde{J} \setminus D)) = (n_j : j \in D, n_j : j \in \tilde{J} \setminus D)$ .

Similarly we use for  $\mathbb{N}^J$ -valued random variables with  $X_t = X(t) = \underline{n} = (n_j : j \in \tilde{J})$  the self explaining abbreviations  $X_D(t) = (n_j : j \in D) \in \mathbb{N}^{|D|}$  and  $X(t) = (X_D(t), X_{\tilde{J} \setminus D}(t))$ .

For a probability space  $(\tilde{\mathbb{E}}, \mathcal{E}, \tilde{\pi})$  and functions  $f, g : (\tilde{\mathbb{E}}, \mathcal{E}) \rightarrow (\mathbb{R}, \mathbb{B})$  we define the inner product of  $f, g$  with respect to  $\tilde{\pi}$ , whenever the following integral exists:  $\langle f, g \rangle_{\tilde{\pi}} = \int_{\tilde{\mathbb{E}}} f(x) \cdot g(x) \tilde{\pi}(dx)$ .

$L^2 := L^2(\tilde{\mathbb{E}}, \tilde{\pi})$  is the space of square integrable functions with respect to  $\tilde{\pi}$ , and  $\|f\|_{\tilde{\pi}} = (\langle f, f \rangle_{\tilde{\pi}})^{1/2}$ .

All random variables occurring are defined on a common underlying probability space  $(\Omega, \mathcal{F}, P)$ .

## 2 Stochastic networks in a random environment

### 2.1 Stochastic networks

A Jackson network [Jac57] consists of  $J$  nodes numbered  $1, \dots, J$ , where indistinguishable customers arrive, are served, possibly at several stations, and eventually depart from the network. The nodes are exponential single servers with state dependent service rates and with an infinite waiting room under first-come-first-served (FCFS) regime. If at node  $j$  there are  $n_j > 0$  customers present, either in service or waiting, then service is provided there at rate  $\mu_j(n_j) > 0$ ; we assume  $\sup\{\mu_j(k) : j \in \{1, \dots, J\}, k \in \mathbb{N}\} < \infty$  and set  $\mu_j(0) := 0$ . All customers follow the same rules. We shall need later on a slight extension of the standard Jackson network models. This is described in terms of an irreducible stochastic routing matrix

$$R = [r_{ij}]_{i,j=0,\dots,J}, \quad (2.1)$$

where the artificial "node 0" represents the source and the sink of all customers. Strict inequality may hold for  $r_{00} \geq 0$ , which means that some arriving customers may be rejected.

Customers arrive in a Poisson stream of intensity  $\lambda > 0$  which is split (independently) according to the first row  $r_0 := (r_{0i} : i = 0, 1, \dots, J)$  of  $R$ . Then at nodes  $j = 1, 2, \dots, J$  we observe independent Poisson- $\lambda_j$  arrival streams with  $\lambda_j = \lambda r_{0j}$ , while a portion  $\lambda_0 = \lambda r_{00}$  of the arriving customers is rejected (lost).

Routing is Markovian, a customer departing from node  $i$  immediately proceeds to node  $j$  with probability  $r_{ij} \geq 0$ , and departs from the network with probability  $r_{j0}$ .

Then the **traffic equations for the admitted customers**

$$\eta_j = \lambda_j + \sum_{i=1}^J \eta_i r_{ij}, \quad j = 1, \dots, J, \quad (2.2)$$

have a unique solution  $(\eta_j : j = 1, \dots, J)$ . Note, that (2.2) only counts for the admitted customers because of  $\lambda_j = \lambda \cdot r_{0j}$ ,  $j = 1, \dots, J$ , and  $\lambda_1 + \dots + \lambda_J = \lambda(1 - r_{00})$ . In case of  $r_{00} = 0$ ,  $R$  is the so-called extended routing matrix of standard Jackson networks, see [DS08] [(3.2)].

Let  $\mathbf{X} = (X_t : t \geq 0)$  denote the vector process recording the joint queue lengths in the network at time  $t$ .  $X_t = (X_1(t), \dots, X_J(t)) \in \mathbb{N}^J$  reads: at time  $t$  there are  $X_j(t)$  customers present at node  $j$ , either in service or waiting. The assumptions put on the system imply that  $\mathbf{X}$  is a strong Markov process on state space  $\mathbb{N}^J$  with generator  $Q^{\mathbf{X}} = (q^{\mathbf{X}}(\underline{n}, \underline{m}) : \underline{m}, \underline{n} \in \mathbb{N}^J)$  which is given for  $g : \mathbb{N}^J \rightarrow \mathbb{R}$  by

$$\begin{aligned} (Q^{\mathbf{X}}g)(\underline{n}) &= \sum_{j=1}^J \lambda_j (g(\underline{n} + e_j) - g(\underline{n})) + \sum_{j=1}^J \mu_j(n_j) r_{j0} (g(\underline{n} - e_j) - g(\underline{n})) \\ &\quad + \sum_{j=1}^J \mu_j(n_j) \sum_{i=1}^J r_{ji} (g(\underline{n} - e_j + e_i) - g(\underline{n})) \end{aligned} \quad (2.3)$$

$Q^{\mathbf{X}}$  is a bounded operator because of  $\inf_{\underline{n} \in \mathbb{N}^J} q^{\mathbf{X}}(\underline{n}, \underline{n}) > -\infty$ . We assume throughout that  $\mathbf{X}$  is ergodic.

For an ergodic network process  $\mathbf{X}$  Jackson's theorem [Jac57] states that the unique steady-state and limiting distribution  $\pi$  on  $\mathbb{N}^J$  is with normalizing constants  $C(j)$  for marginal distributions of  $\mathbf{X}$

$$\pi(\underline{n}) = \pi(n_1, \dots, n_J) = \prod_{j=1}^J \left( C(j)^{-1} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right), \quad \underline{n} = (n_1, \dots, n_J) \in \mathbb{N}^J, \quad (2.4)$$

## 2.2 Breakdown-repair processes

We are interested in stochastic networks, where nodes due to external environment influences can breakdown and are repaired periodically. A common situation is that the breakdown-repair process is Markov of its own, and the network reacts on the perturbations driven by the random environment. To describe these Markovian processes we consider a set of  $J$  stations or devices (nodes) numbered  $1, \dots, J$ . Stations are unreliable, break down randomly and are repaired thereafter. Repair time is random as well. We assume that the availability status of the system can be described by a homogeneous Markov process

$\mathbf{Y} = (Y(t) : t \geq 0), \quad Y(t) : (\Omega, \mathcal{F}, P) \rightarrow (2^{\tilde{J}}, \mathcal{P}(2^{\tilde{J}})).$

$Y(t) = D$  indicates that at time  $t \geq 0$  the stations in  $D \subseteq \tilde{J}$  are down and under repair, while stations in  $\tilde{J} \setminus D \subseteq \tilde{J}$  are functioning ("are up"). The transition rates (breakdown and repair intensities) of  $\mathbf{Y}$  are

**Definition 2.1** Take any pair of functions  $A : 2^{\tilde{J}} \rightarrow [0, \infty)$  and  $B : 2^{\tilde{J}} \rightarrow [0, \infty)$ , subject to  $A(\emptyset) = 1$  and  $B(\emptyset) = 1$  and for  $D, I, H \subseteq \tilde{J}$  (we set  $0/0 = 0$  and  $1/0 = \infty$ ) subject to

$$\frac{A(I)}{A(D)} < \infty \quad \forall D \subset I \subseteq \tilde{J} \quad \text{and} \quad \frac{B(D)}{B(H)} < \infty \quad \forall H \subset D \subseteq \tilde{J}.$$

With these functions define breakdown and repair rates as follows:

$$q^{\mathbf{Y}}(D, I) = \frac{A(I)}{A(D)}, \quad D \subset I \subseteq \tilde{J},$$

for breakdowns of nodes in non-empty set  $I \setminus D$  if nodes in  $D$  are already down, and

$$q^{\mathbf{Y}}(D, H) = \frac{B(D)}{B(H)}, \quad H \subset D \subseteq \tilde{J},$$

for finishing repair of nodes in non-empty set  $D \setminus H$  if nodes in  $D$  are under repair. For all other pairs  $G, H \subseteq \tilde{J}, G \neq H$ , we set  $q^{\mathbf{Y}}(G, H) = 0$ , and for all  $D \subseteq \tilde{J}$  we set  $q^{\mathbf{Y}}(D, D) = -\sum_{H \subseteq \tilde{J}, H \neq D} q^{\mathbf{Y}}(D, H)$ .

The generator  $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K, L) : K, L \subseteq \tilde{J})$  of  $\mathbf{Y}$  is defined for real functions  $f : 2^{\tilde{J}} \rightarrow \mathbb{R}$ , by

$$(Q^{\mathbf{Y}}f)(D) = \sum_{H \subset D} \frac{B(D)}{B(H)}(f(H) - f(D)) + \sum_{I \supset D} \frac{A(I)}{A(D)}(f(I) - f(D)) \quad (2.5)$$

By evaluation of the standard local balance equations it can be seen that  $\mathbf{Y}$  is reversible with respect to the probability measure (with normalization constant  $\hat{C}^{-1}$ )

$$\hat{\pi} := \left( \hat{\pi}(D) := \hat{C}^{-1} \frac{A(D)}{B(D)}, \quad D \in 2^{\tilde{J}} \right). \quad (2.6)$$

## 2.3 Rerouting

The network process and the breakdown-repair process (availability process) interact and we have to fix rules for the interaction regime. The general rule is:

- (1) Whenever a station is broken down and under repair, service is interrupted and the customers present there are frozen, while new customers are not admitted to this station.
- (2) Therefore we have to define a new routing mechanism. Examples of how to do this to obtain explicit steady states can be found in [SD03][Sections 5, 6]. We describe an abstract "rerouting scheme", which encompasses the three schemes described there.

**Assumption 2.2** REROUTING SCHEMES IN OPEN NETWORKS. Consider a Jackson network with routing matrix (2.1) (with  $(r_{00} \geq 0)$ ) and traffic equations for the admitted customers (2.2) where  $\lambda_j = \lambda r_{0j}$ .

When nodes in  $D \subseteq \tilde{J}$  are down, routing is restricted to  $\tilde{J}_0 \setminus D$  and determined by some routing matrix

$$R^D = [r_{ij}^D]_{i,j \in \tilde{J}_0 \setminus D}. \quad (2.7)$$

The associated traffic equations for the admitted customers similar to (2.2) are

$$\eta_j^D = \lambda_j^D + \sum_{i \in \tilde{J} \setminus D} \eta_i^D r_{ij}^D, \quad j \in \tilde{J} \setminus D, \quad \text{with } \lambda_j^D := \lambda \cdot r_{0,j}^D, \quad j \in \tilde{J} \setminus D, \quad (2.8)$$

and are assumed to be solved by

$$\eta_j^D = \eta_j, \quad j \in \tilde{J} \setminus D, \quad (2.9)$$

where the  $\eta_j$  are from the solution of (2.2). We set  $R^\emptyset := R$  and, if necessary,  $\eta_j^\emptyset := \eta_j$ . For the rerouting scheme with nodes in set  $D$  broken down  $\lambda_0^D := \lambda \cdot r_{0,0}^D$  is the new rejection rate.

**Lemma 2.3** If  $(\eta_j^D, j \in \tilde{J} \setminus D)$  solves the traffic equations (2.8) for the admitted customers, when nodes in  $D$  are broken down and rerouting is according to Assumption 2.2, then with  $\eta_0^D := \lambda$  the vector  $(\eta_j^D, j \in \tilde{J}_0 \setminus D)$  solves the equation  $x = x \cdot R^D$ .

Proof. (2.8) can be written as  $\eta_j^D = \lambda \cdot r_{0j}^D + \sum_{i \in \tilde{J} \setminus D} \eta_i^D r_{ij}^D$ ,  $j \in \tilde{J} \setminus D$ . Summing over  $j \in \tilde{J} \setminus D$  yields  $\lambda(1 - r_{00}^D) = \sum_{i \in \tilde{J} \setminus D} \eta_i^D r_{i0}^D$ , which is the missing equation of  $x = x \cdot R^D$ , with the required solution inserted.  $\square$

**Remark 2.4** (i) For technical simplifications we introduce the normalized probability vector  $\xi^D$  associated with values  $\eta_j^D$ :  $\xi_j^D = \eta_j^D / (\sum_{k=0}^J \eta_k^D)$ ,  $j \in \tilde{J}_0 \setminus D$ . Similarly we define  $\xi = \xi^\emptyset$  associated with the  $\eta_j = \eta_j^\emptyset$ .

(ii) Prescribing rerouting by (2.8) is not constructive, but is not necessary for our main applications. A detailed description of rerouting schemes which fulfill Assumption 2.2 is given in [Sau06][Section 2].

(iii) When considering rerouting schemes which are used in the literature it may happen that the rerouting chain on certain subsets  $\tilde{J} \setminus D$  is not irreducible, for details see [Sau06][Proof of Theorem 1.2.29]. This makes the computations more involved, but leads to the same results as those we shall present below.

**Example 2.5** The most common rerouting schemes found in the literature which lead to explicit stationary distributions of the network processes are (for more details see [SD03][Sections 5, 6] and [DS13][Section 2.3])

(i) RS–RD WITH REVERSIBLE ROUTING. This applies only for reversible  $R$ . When a customer tries to visit a down node he is rejected, stays at his node for another service and tries again with newly selected destination.

- (ii) **STALLING.** Whenever any node breaks down all services and arrivals are interrupted and resumed only when all nodes are up again.
- (iii) **SKIPPING.** Whenever a customer wants to visit a broken down node he is not allowed to settle down there and has to jump forward according to  $R$  until he reaches an up-node or leaves the network.

## 2.4 Networks with breakdown and repair: Product formula

A Markovian state process for an unreliable Jackson network requires that the state space  $\mathbb{N}^J$  of the Jackson network process  $\mathbf{X}$  is supplemented by a coordinate  $\mathbf{Y}$  which indicates the set of broken down stations. Operating on these states we define a Markov process  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  describing the degradable network with state space  $\tilde{\mathbb{E}} = 2^J \times \mathbb{N}^J$ . States are  $\mathbf{n} = (D, \underline{n}) = (D, n_1, n_2, \dots, n_J) \in \tilde{\mathbb{E}}$ , where the first coordinate in  $\mathbf{n}$  we call the availability coordinate. The interpretation is:  $D$  is the set of servers in *down status*. At node  $i \in D$  there are  $n_i$  customers waiting for server being repaired. We denote by  $\mathcal{E} := 2^{\tilde{\mathbb{E}}}$ .

**Definition 2.6** THE UNRELIABLE JACKSON NETWORK PROCESS is the Markov process  $\mathbf{Z} = (Z(t), t \geq 0)$  defined by the infinitesimal generator (transition intensity matrix)  $Q^{\mathbf{Z}} = (q^{\mathbf{Z}}(\mathbf{n}, \mathbf{n}') : \mathbf{n}, \mathbf{n}' \in \tilde{\mathbb{E}})$  via

$$\begin{aligned} (Q^{\mathbf{Z}}f)(D, n_1, n_2, \dots, n_J) &= \sum_{j \in \tilde{J} \setminus D} \lambda r_{0j}^D (f(D, \underline{n} + e_j) - f(\mathbf{n})) + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) r_{j0}^D (f(D, \underline{n} - e_j) - f(\mathbf{n})) \\ &\quad + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) \sum_{i \in \tilde{J} \setminus D} r_{ji}^D (f(D, \underline{n} - e_j + e_i) - f(\mathbf{n})) \\ &\quad + \sum_{H \subset D} \frac{B(D)}{B(H)} (f(H, \underline{n}) - f(\mathbf{n})) + \sum_{I \supset D} \frac{A(I)}{A(D)} (f(I, \underline{n}) - f(\mathbf{n})). \end{aligned} \quad (2.10)$$

**Theorem 2.7** PRODUCT FORM FOR JACKSON NETWORKS WITH BREAKDOWN AND REPAIR. [SD03],

[Sau06][Theorem 2.4.1] Under the Assumption 2.2, if  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  is ergodic then the steady state is with  $\pi$  from (2.4) and  $\hat{\pi}$  from (2.6) of the product form: For  $(D, n_1, \dots, n_J) \in \tilde{\mathbb{E}}$

$$\tilde{\pi}(D, n_1, n_2, \dots, n_J) = \hat{\pi}(D) \cdot \pi(n_1, n_2, \dots, n_J) = \hat{C}^{-1} \frac{A(D)}{B(D)} \cdot \prod_{j=1}^J \left( C_j^{-1} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)} \right). \quad (2.11)$$

Here  $\eta = (\eta_1, \dots, \eta_J)$  is the solution of the traffic equation (2.2) for admitted customers when all nodes are up,  $C_j$  is the normalization constant for the local queue length process at node  $j$ . We denote  $C = \prod_{j=1}^J C_j$ .

(2.11) is proved in [SD03] for more general breakdown and repair schemes: Breakdown and repair rates may depend on the load (queue lengths) of nodes. The question whether in this framework results similar to those in the following sections can be derived is still open and part of our ongoing research.

Theorem 2.7 is not covered by the results for networks in a random environment in [Zhu94] and [Eco05]. In both papers it assumed that under different environment states the ratio "local arrival rate/local service rate" is independent of the environment status. This is obviously not the case in our systems.



### 3 One step correlation

Recall  $\lambda_j = \lambda \cdot r_{0j}$ ,  $j = 1, \dots, J$ , and  $\lambda_1 + \dots + \lambda_J = \lambda(1 - r_{00})$  and that we therefore consider only admitted customers even if all nodes are up. We will not mention this further. For the network process  $\mathbf{Z}$  with generator  $Q^{\mathbf{Z}}$  and stationary distribution  $\tilde{\pi}$  consider *one step* correlation expressions

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}}. \quad (3.1)$$

If  $f = g$ , then (3.1) is (the negative of) a quadratic form, because  $-Q^{\mathbf{Z}}$  is positive definite. (3.1) occurs in the definition of Cheeger's constant because division of (3.1) with  $f = g$  by  $\langle f, f \rangle_{\tilde{\pi}}$  yields Rayleigh quotients. It also occurs in the definition of the corresponding Dirichlet form. This is helpful to bound the second largest eigenvalue of  $Q^{\mathbf{Z}}$  and to prove the Poincare inequality for the corresponding Markovian process, see e.g., [Che04]. Furthermore, (3.1) can be utilized to determine the asymptotic variance of costs or performance measures associated with Markovian network processes and to compare the asymptotic variances of two such related processes. It is possible to compare the correlations for  $\mathbf{Z}$  with that of the related process  $\mathbf{Z}'$  with the same stationary distribution  $\tilde{\pi}$ , using  $\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}}$ , which will be given explicitly in Section 3.2. Because we are dealing with processes having bounded generators, properties connected with (3.1) can be turned into properties of  $\langle f, Id + \varepsilon Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} = E_{\tilde{\pi}}(f(Z_0)g(Z_\tau))$  where  $\varepsilon > 0$  is sufficiently small such that  $Id + \varepsilon Q^{\mathbf{Z}}$  is a stochastic matrix and  $\tau$  is exponentially distributed. This enables one to directly apply discrete time methods to characterize properties of continuous time processes.

#### 3.1 Correlation formulas

Due to the product form steady state distribution of  $\mathbf{Z}$  the one step correlation  $\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}}$  splits immediately into two terms having an intuitive interpretation: The sum of weighted one step conditional correlations

(i) of the environment process  $Y$ , and (ii) of network processes, which for a fixed time point seem to behave conditionally independent of the environment.

As will be seen, it is illuminating to define for all  $D \subseteq \tilde{J}$  the generators  $Q^{\mathbf{X}_{\tilde{J} \setminus D}}$  of "synthetic subnetworks" on node set  $\tilde{J} \setminus D$  with overall arrival rate  $\lambda$ , service rates from Definition 2.6, and routing matrix  $R^D$ .

The proofs of Propositions 3.1 and 3.2 are postponed to Section 6.

**Proposition 3.1** SPLITTING FORMULA. *For unreliable Jackson network processes  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  the one step correlations splits as follows*

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} = & \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \left\{ \sum_{D \subseteq \tilde{J}} \hat{\pi}(D) f(D, \underline{n}) (Q^{\mathbf{Y}} g(\cdot, \underline{n})) (D) \right\} + \sum_{D \subseteq \tilde{J}} \hat{\pi}(D) \sum_{\underline{n}_D \in \mathbb{N}^{|D|}} \pi_D(\underline{n}_D) \\ & \cdot \left\{ \sum_{\underline{n}_{\tilde{J} \setminus D} \in \mathbb{N}^{J-|D|}} \pi_{\tilde{J} \setminus D}(\underline{n}_{\tilde{J} \setminus D}) f(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D})) \left( Q^{\mathbf{X}_{\tilde{J} \setminus D}} g(D, (\underline{n}_D, (\cdot)_{\tilde{J} \setminus D})) \right) (\underline{n}_{\tilde{J} \setminus D}) \right\} \end{aligned}$$



The next correlation formula will yield remarkable simplifications when used for differences.

**Proposition 3.2** ONE-STEP CORRELATION FORMULA. *For unreliable Jackson network processes  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  with  $\xi^D$  the probability solution of  $x^D = x^D \cdot R^D$  (when nodes in  $D$  are down) holds*

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} = & \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \\ & + \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j \in \tilde{J}_0 \setminus D} \sum_{i \in \tilde{J}_0 \setminus D} \xi_j^D r_{ji}^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) \right\} \\ & - \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) + \lambda + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) \right\}. \end{aligned} \quad (3.2)$$

### 3.2 Comparison of one step correlations

The following formulas for differences of one step correlations will give additional insight into various properties of networks, for example to speed of convergence or asymptotic variance. They display how, e.g., the routing and the breakdown and repair affects correlations in networks.

**Theorem 3.3** CHANGES OF ROUTING BEHAVIOUR OF THE CUSTOMERS. *Suppose  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  is an ergodic unreliable Jackson network process with a routing matrix  $R$  and  $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$  is another Jackson network process having the same arrival and service intensities and failure-repair rates but with routing matrix  $R' = [r'_{ij}]_{i,j=0,1,\dots,J}$ , such that the solutions of the traffic equation derived from  $R$  and for  $R'$  coincide (denoted by  $\eta$ ). Assume that both networks follow some rerouting mechanism for which the Assumption 2.2 holds. Then for arbitrary real functions  $f, g \in L^2$*

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}} = E_{\tilde{\pi}} \left[ \frac{\lambda}{\xi_0^{Y_t}} \left( \text{tr}(W^{g,f}(Y_t, X_t) \cdot \text{diag}(\xi^{Y_t}) \cdot (R^{Y_t} - R'^{Y_t})) \right) \right],$$

where  $\xi^D$  is the probability solution of  $x^D = x^D \cdot R^D$ ,  $\text{tr}(A)$  denotes trace of  $A$ , and

$$W^{g,f}(D, \underline{n}) = [g(D, \underline{n} + e_i) f(D, \underline{n} + e_j)]_{i,j \in \tilde{J}_0 \setminus D}.$$

Proof. Because the external arrival streams are the same, and the traffic equations have the same solution  $\eta$ , and the rerouting mechanisms have property (2.9), for any availability status  $D$  the rerouting schemes on  $\tilde{J} \setminus D$  have the same solution of the traffic equation. It follows from Lemma 2.3 that for all  $D$  the probability solution of the equations  $x^D = x^D \cdot R^D$  and  $x^D = x^D \cdot R'^D$  are in both systems the same. Because of  $q^{\mathbf{Y}} = q^{\mathbf{Y}'}$  we have from Proposition

### 3.2 the reduction

$$\begin{aligned}
\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}} &= \sum_{D \subseteq \{1, \dots, J\}} C^{-1} \hat{\pi}(D) \sum_{\underline{n}_D \in \mathbb{N}^{|D|}} \prod_{\ell \in D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) \\
&\cdot \sum_{\underline{n}_{\tilde{J} \setminus D} \in \mathbb{N}^{J-|D|}} \prod_{\ell \in \tilde{J} \setminus D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) \left[ \frac{\lambda}{\xi_0^D} \left\{ \sum_{j \in \tilde{J}_0 \setminus D} \sum_{i \in \tilde{J}_0 \setminus D} \xi_j^D r_{ji}^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) \right\} \right. \\
&\quad \left. - \frac{\lambda}{\xi_0^D} \left\{ \sum_{j \in \tilde{J}_0 \setminus D} \sum_{i \in \tilde{J}_0 \setminus D} \xi_j^D r'_{ji}^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) \right\} \right]
\end{aligned}$$

We interpret in the last two lines for fixed  $D$  and  $\underline{n}_D$  and  $i, j \in \tilde{J}_0 \setminus D$  the expressions

$$f(D, \underline{n} + e_j) =: f(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D} + e_j)) \quad \text{and} \quad g(D, \underline{n} + e_i) =: g(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D} + e_i))$$

as functions of  $\underline{n}_{\tilde{J} \setminus D}$  only, and see that the resulting expressions have exactly the structure of the functions dealt with in Proposition 4.1 of [DS08]. After renormalization of the densities  $\prod_{\ell \in \tilde{J} \setminus D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right)$ , which in fact results in conditioning on  $\{Y(t) = D, X_D(t) = \underline{n}_D\}$ , we obtain  $\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}}$

$$\begin{aligned}
&= \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n}_D \in \mathbb{N}^{|D|}} P(\{Y(t) = D, X_D(t) = \underline{n}_D\}) \\
&\quad \cdot E_{\tilde{\pi}} \left[ \frac{\lambda}{\xi_0^D} \left( \text{tr}(W^{g,f}(D, (\underline{n}_D, X_{\tilde{J}_0 \setminus D})) \cdot \text{diag} \xi^D \cdot (R^D - R'^D)) \right) | \{Y(t) = D, X_D(t) = \underline{n}_D\} \right],
\end{aligned}$$

and deconditioning eventually finishes the proof.  $\square$

**Theorem 3.4** CHANGES OF BREAKDOWN AND REPAIR MECHANISMS. *Suppose  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  is an ergodic unreliable Jackson network process with a routing matrix  $R = [r_{ij}]_{i,j=0,1,\dots,J}$  and  $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$  is another Jackson network process having the same arrival and service intensities, and with the same routing regime, described by  $R$  and rerouting fulfilling Assumption 2.2.*

*The breakdown-repair process for  $\mathbf{Z}$  is given in Definition 2.1 and is for  $\mathbf{Z}'$  defined similarly via  $A', B' : 2^{\tilde{J}} \rightarrow [0, \infty)$ , subject to the restrictions indicated there. Then the breakdown and repair rates for  $\mathbf{Y}'$  are:*

$$q^{\mathbf{Y}'}(D, I) = \frac{A'(I)}{A'(D)}, \quad D \subset I \subseteq \tilde{J}, \quad \text{and} \quad q^{\mathbf{Y}'}(D, H) = \frac{B'(D)}{B'(H)}, \quad H \subset D \subseteq \tilde{J}, .$$

*The processes  $\mathbf{Y}$  and  $\mathbf{Y}'$  are Markov with generators  $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K, L) : K, L \subseteq \tilde{J})$  of  $\mathbf{Y}$  and  $Q^{\mathbf{Y}'} = (q^{\mathbf{Y}'}(K, L) : K, L \subseteq \tilde{J})$  of  $\mathbf{Y}'$  as defined in (2.5) for  $\mathbf{Y}$  and similar for  $\mathbf{Y}'$ .*

*Assume that the stationary distributions of  $\mathbf{Y}$  and  $\mathbf{Y}'$  are identical, denoted by*

$$\hat{\pi} := \left( \hat{\pi}(D) := \hat{C}^{-1} \frac{A(D)}{B(D)} = \hat{C}'^{-1} \frac{A'(D)}{B'(D)}, \quad D \in 2^{\tilde{J}} \right).$$

Then for arbitrary real functions  $f, g : \tilde{\mathbb{E}} \rightarrow \mathbb{R}$  holds

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\hat{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\hat{\pi}} &= E_{\pi} \left[ \langle f(\circ, X_t), Q^{\mathbf{Y}} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} - \langle f(\circ, X_t), Q^{\mathbf{Y}'} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right] = \\ &= E_{\pi} \left[ \langle f(\circ, X_t), (Q^{\mathbf{Y}} - Q^{\mathbf{Y}'}) g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right]. \end{aligned}$$

Proof. Interchanging summations, regrouping terms, and exploiting the product form structure of the state distributions (which are identical for  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  and  $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$ ) in (3.2) we obtain

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\hat{\pi}} &= \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \\ &\quad - \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right\} \\ &\quad + \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j \in \tilde{J}_0 \setminus D} \sum_{i \in \tilde{J}_0 \setminus D} \xi_j^D r_{ji}^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) \right\} \\ &\quad - \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) f(D, \underline{n}) g(D, \underline{n}) \left\{ \lambda + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) \right\}. \end{aligned}$$

For  $\langle f, Q^{\mathbf{Z}'} g \rangle_{\hat{\pi}}$  the last two lines in the respective formula are identical to those in the displayed formula. The difference therefore is  $\langle f, Q^{\mathbf{Z}} g \rangle_{\hat{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\hat{\pi}} =$

$$\begin{aligned} &= \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right. \\ &\quad - \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right\} \\ &\quad - \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \left\{ \sum_{H \subset D} q^{\mathbf{Y}'}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}'}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \\ &\quad \left. + \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}'}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}'}(D, I) \right\} \right\} \end{aligned}$$

For fixed  $\underline{n}$  we interpret  $f(D, \underline{n})$  and  $g(D, \underline{n})$  as functions of  $D$  parametrized by  $X_t(\omega) = \underline{n}$ . This leads to

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\hat{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\hat{\pi}} &= \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \left[ \langle f(\circ, \underline{n}), Q^{\mathbf{Y}} g(\cdot, \underline{n})(\circ) \rangle_{\hat{\pi}} - \langle f(\circ, \underline{n}), Q^{\mathbf{Y}'} g(\cdot, \underline{n})(\circ) \rangle_{\hat{\pi}} \right] \\ &= E_{\pi} \left[ \langle f(\circ, X_t), Q^{\mathbf{Y}} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} - \langle f(\circ, X_t), Q^{\mathbf{Y}'} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right]. \end{aligned} \tag{3.3}$$

□

Theorems 3.3 and 3.4 state a reduction of complexity: Comparing operators and resulting one step correlations via functions on infinite space is reduced to comparing matrix operators via functions on finite space. The theorems are valid for all square integrable functions  $f, g$  on  $\tilde{\mathbb{E}}$ . This opens the way to compare correlations for multidimensional marginals in time of the network processes with unreliable nodes according to concordance ordering utilizing the abstract setting for Markov processes of [DS08][Theorem 5.2].

For a concise notation we introduce the standard difference operators for functions on  $\mathbb{N}^J$ . For all  $f \in L^2$  and all  $j = 0, 1, \dots, J$ , we define (recall  $e_0$  is the zero vector)

$$\mathcal{D}_j f : \tilde{\mathbb{E}} \rightarrow \mathbb{R}, \quad (D, \underline{n}) \rightarrow \mathcal{D}_j f(D, \underline{n}) := f(D, \underline{n} + e_j) - f(D, \underline{n}),$$

and

$$\mathcal{D}f : \tilde{\mathbb{E}} \rightarrow \mathbb{R}^{J+1-|D|}, \quad (D, \underline{n}) \rightarrow (\mathcal{D}_j f(D, \underline{n}), j \in \tilde{J}_0 \setminus D).$$

That way we can treat  $\mathcal{D}f(D, \underline{n})$  as a vector of the dimension corresponding to the size of  $D$ , and the corresponding routing matrices  $R^D$  as operators on it. Moreover, it is possible to consider the corresponding scalar products generated by invariant vectors  $\xi^D$ , and write the formula for the difference of one step correlations (which are scalar products with respect to invariant measures for the network process  $\tilde{\pi}$ ) in terms of scalar products with respect to invariant measures  $\xi^D$  for the routing processes.

**Corollary 3.5** *For unreliable Jackson network processes  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  with  $\xi^D$  the probability solution of the equation  $x^D = x^D \cdot R^D$  and  $\underline{\mu}_D = \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j)$  the total service rate for nodes which are up, we have*

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} &= E_{\tilde{\pi}} \left[ \frac{\lambda}{\xi_{\xi_0}^{Y_t}} \langle (\mathcal{D} + Id)f(Y_t, X_t), R^{Y_t}(\mathcal{D} + Id)g(Y_t, X_t) \rangle_{\xi^{Y_t}} \right] \\ &+ E_{\pi} [\langle f(\circ, X_t), Q^{\mathbf{Y}} g(\cdot, X_t)(\circ) \rangle_{\tilde{\pi}}] - E_{\tilde{\pi}} [f(Y_t, X_t)g(Y_t, X_t)(\lambda + \underline{\mu}_{Y_t})]. \end{aligned}$$

Proof. Take Proposition 3.2 (3.2), condition as in Theorem 3.3, and insert the suitable difference operators. □

We can now reformulate the result of Theorem 3.3 in a more compact form which immediately relates our results to methods dealt with in optimizing MCMC simulation.

**Corollary 3.6** *For unreliable Jackson network processes  $\mathbf{Z}, \mathbf{Z}'$  as in Theorem 3.3 we have*

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}} = E_{\tilde{\pi}} \left[ \frac{\lambda}{\xi_{\xi_0}^{Y_t}} \left\langle (\mathcal{D} + Id)f(Y_t, X_t), (R^{Y_t} - R'^{Y_t})(\mathcal{D} + Id)g(Y_t, X_t) \right\rangle_{\xi^{Y_t}} \right].$$

## 4 Applications

The proofs of the applications below follow the ideas which are used to prove the theorems in Section 3.2. We therefore give only hints to perform the proofs here.

## 4.1 Comparison of spectral gaps

Let  $\mathbf{Z}$  be a continuous time homogeneous ergodic Markov process with stationary probability  $\tilde{\pi}$  and generator  $Q^{\mathbf{Z}}$ . Let  $\tilde{\pi}(f) = \int_{\mathbb{E}} f(x) \tilde{\pi}(dx)$ . The spectral gap of  $\mathbf{Z}$ , resp.  $Q^{\mathbf{Z}}$  is

$$\text{Gap}(Q^{\mathbf{Z}}) = \inf\{\langle f, -Q^{\mathbf{Z}}f \rangle_{\tilde{\pi}} : f \in L^2(\mathbb{E}, \tilde{\pi}), \tilde{\pi}(f) = 0, \langle f, f \rangle_{\tilde{\pi}} = 1\}.$$

The spectral gap determines for  $\mathbf{Z}$  the speed of convergence to equilibrium  $\tilde{\pi}$  in  $L^2(\mathbb{E}, \tilde{\pi})$  with norm  $\|\cdot\|_{\tilde{\pi}}$ :  $\text{Gap}(Q^{\mathbf{Z}})$  is the largest number  $\Delta$  such that for the transition semigroup  $P = (P_t : t \geq 0)$  of  $\mathbf{Z}$  holds

$$\|P_t f - \tilde{\pi}(f)\|_{\tilde{\pi}} \leq e^{-\Delta t} \|f - \tilde{\pi}(f)\|_{\tilde{\pi}} \quad \forall f \in L^2(\mathbb{E}, \tilde{\pi}).$$

It should be noted that one has to be careful which class of functions is used for the definition of spectral gap. For a discussion and more references see the introduction of [LS14].

We utilize the following orderings to compare routings, failure processes and then correlations, see [Pes73].

**Definition 4.1** Let  $R = [r_{ij}]$  and  $R' = [r'_{ij}]$  be transition matrices on a finite set  $\mathbb{E}$  such that  $\xi R = \xi R' = \xi$  for a probability vector  $\xi$ .

$R'$  is smaller than  $R$  in the positive semidefinite order,  $R' \prec_{pd} R$ , if the matrix  $R - R'$  is positive semidefinite.

$R'$  is smaller than  $R$  in the Peskun order,  $R' \prec_P R$ , if for all  $j, i \in \mathbb{E}$  with  $i \neq j$  holds  $r'_{ji} \leq r_{ji}$ .

Peskun used the latter order to compare reversible transition matrices with the same stationary distribution and their asymptotic variance, and Tierney [Tie98] has shown (in a more general setting, i.e. using operators rather than matrices) that the main property used in the proof of Peskun, namely that " $R \prec_P R'$  implies  $R' \prec_{pd} R$ ", holds without reversibility assumptions.

**Example 4.2** For any transition matrix  $R = [r_{ij}]$  and  $R' = Id$  (of the same dimension as  $R$ ) holds  $R' \prec_P R$ , so the family of transition matrices of fixed dimension has a (unique) minimal element. Therefore STALLING from Example 2.5 is an extremal (re-)routing scheme, because it utilizes  $Id$  in case of any breakdown.

**Proposition 4.3** Consider two ergodic unreliable Jackson networks with state processes  $\mathbf{Z}$  and  $\mathbf{Z}'$  and with the same arrival and service intensities, and the same failure-repair rates. Assume that the equations  $x = x \cdot R$  and  $x = x \cdot R'$  have the same normalized solution  $\xi$ , and the Assumption 2.2 holds, i.e. both networks follow some rerouting mechanism according to (2.8) with the property (2.9).

If  $R^D \prec_{pd} R'^D$  for all  $D$  (also for  $D = \emptyset$ ), then

$$\text{Gap}(Q^{\mathbf{Z}'}) \leq \text{Gap}(Q^{\mathbf{Z}}).$$

Proof. From Corollary 3.6 we have for all  $f \in L^2$

$$\langle f, -Q^{\mathbf{Z}}f \rangle_{\tilde{\pi}} - \langle f, -Q^{\mathbf{Z}'}f \rangle_{\tilde{\pi}} = E_{\tilde{\pi}} \left[ \frac{\lambda}{\xi^{Y_t}} \langle (\mathcal{D} + Id)f(Y_t, X_t), (R'^{Y_t} - R^{Y_t})(\mathcal{D} + Id)f(Y_t, X_t) \rangle_{\xi^{Y_t}} \right],$$

and from the product formula we rewrite this formula as

$$\begin{aligned} \langle f, -Q^{\mathbf{Z}} f \rangle_{\tilde{\pi}} - \langle f, -Q^{\mathbf{Z}'} f \rangle_{\tilde{\pi}} &= E_{\pi} E_{\hat{\pi}} \left[ \frac{\lambda}{\xi_0^{Y_t}} \langle (\mathcal{D} + Id) f(Y_t, X_t), (R^{Y_t} - R^{Y_t})(\mathcal{D} + Id) f(Y_t, X_t) \rangle_{\xi^{Y_t}} \right] = \\ &= E_{\pi} \sum_D \hat{\pi}(D) \left[ \frac{\lambda}{\xi_0^D} \langle (\mathcal{D} + Id) f(D, X_t), (R'^D - R^D)(\mathcal{D} + Id) f(D, X_t) \rangle_{\xi^D} \right]. \end{aligned}$$

From  $\prec_{pd}$  ordering of routings for all  $f \in L^2$  therefore  $\langle f, -Q^{\mathbf{Z}} f \rangle_{\tilde{\pi}} \geq \langle f, -Q^{\mathbf{Z}'} f \rangle_{\tilde{\pi}}$  holds.  $\square$

**Remark 4.4** *If  $R \prec_{pd} R'$  and the rerouting for  $\mathbf{Z}'$  is by stalling, then by the extremal property of  $Id$  for  $\prec_P$  the assumptions of Proposition 4.3 are fulfilled.*

Computing spectral gaps for Markov processes with multidimensional state space is challenging, in many cases nearly impossible. Exceptions are multidimensional independent birth-death processes, because for birth-death processes explicit results are known, see e.g. [Doo02], and Liggett has proved that the gap of independent processes is the minimum of the spectral gap of the marginal processes [Lig89][Theorem 6.2]. We will show that the gap of the joint queue length network process  $\mathbf{Z}$  (with unreliable nodes) can be bounded from below by the gap of a network process consisting of identical breakdown-repair process and related multidimensional birth-death process with **conditionally independent** components.

**Proposition 4.5** *Consider an ergodic Jackson network process  $\mathbf{Z}$  with unreliable servers as in Theorem 2.7. Assume that for all  $D \subseteq \{1, 2, \dots, J\}$  with  $\hat{\pi}(D) > 0$  the routing matrix  $R^D = [r_{ij}^D]_{i,j \in \tilde{J}_0 \setminus D}$  has strict positive entrance and departure probabilities ( $r_{0i}^D > 0, r_{i0}^D > 0$ ) for every node  $i \in \tilde{J} \setminus D$ .*

*Assume further that for all  $D \subseteq \{1, 2, \dots, J\}$  the rerouting  $R^D$  fulfills overall balance for all network nodes which are up, i.e.*

$$\eta_j^D \sum_{i \in \tilde{J} \setminus D} r_{ji}^D = \sum_{i \in \tilde{J} \setminus D} \eta_i^D r_{ij}^D, \quad \forall j \in \tilde{J} \setminus D. \quad (4.1)$$

*Then there exists an ergodic Jackson network process process  $\mathbf{Z}'$  with unreliable servers as in Theorem 2.7 with the same stationary distribution  $\tilde{\pi}$  as  $\mathbf{Z}$ , such that*

$$\text{Gap}(Q^{\mathbf{Z}'}) \leq \text{Gap}(Q^{\mathbf{Z}}).$$

*The nodes of  $\mathbf{Z}'$  are perturbed by a common breakdown-repair regime identical to that of  $\mathbf{Z}$ , and for any given set  $D$  of broken down nodes the joint network process on  $\tilde{J} \setminus D$  consists of conditionally independent birth-death processes, and the coordinate birth and death processes on the  $i$ -th coordinate have birth rate  $\lambda r_{0i}^D$  and state dependent death rate  $\mu_i(n_i) r_{i0}^D$ .*

Proof. In order to obtain a lower bound for the spectral gap using birth and death processes the idea is to allow in the comparison network for each node  $i$ , which is up, that any customer who enters node  $i$  from the external source after being served only to feed back (possibly iteratively) to node  $i$  or to depart from the network. This results in updating the service

rates suitably. Recall that the original network processes and the constructed comparison processes are additionally perturbed by the same failure mechanism.

Consider the situation when nodes in  $D$  are down. Directly from the generator of the network process  $\mathbf{Z}$  in (2.10), it is clear that after reducing movements inside the network, and allowing only for movements into the network from outside, or from the network into outside, or feedback, we get as long as the reliability level  $D$  does not change transitions for changing the queue lengths which look identical as those of the generator of independent birth and death processes such that on the  $i$ -th coordinate the birth rate equals  $\lambda r_{0i}^D$  and the state dependent death rate equals  $\mu_i(n_i) r_{i0}^D, i \in \tilde{J} \setminus D$ .

Now, in order to be able to apply a formula for differences of one step correlations we have to show on every reliability level  $D$  that such a modification is possible within a class of networks with extended routings having the same stationary solution. For this reason we need the assumption on overall balance (4.1).

More precisely, we define  $R'^D$  by  $r'_{i0}^D = r_{i0}^D, r'_{0i}^D = r_{0i}^D$ , for all  $i, r'_{ij}^D = 0$  for  $j \neq i, i, j \in \tilde{J} \setminus D$ , and  $r'_{ii}^D = 1 - r_{i0}^D$  for  $i \in \tilde{J} \setminus D$ . With the routing  $R'^D$ , the network process  $\mathbf{Z}'$  (when nodes in  $D$  are down) develops as a vector of independent birth and death processes for the up nodes in  $\tilde{J} \setminus D$ .

For  $j \in \{1, \dots, J\}$  let  $\eta_j'^D$  be the solution of the traffic equations for  $R'^D$ . We have directly

$$\eta_j'^D = \lambda r_{0j}^D + \eta_j'^D r_{jj}'^D, \quad j \in \tilde{J} \setminus D, \quad (4.2)$$

and the solution of this system is uniquely defined. We show that (4.2) is solved (for each  $D$ ) by  $(\eta_j^D, j \in \tilde{J} \setminus D)$  as well. Inserting this into (4.2) we obtain with  $r_{jj}'^D = 1 - r_{j0}^D$  for  $j \in \tilde{J} \setminus D$

$$\eta_j^D = \lambda r_{0j}^D + \eta_j^D (1 - r_{j0}^D) = \lambda r_{0j}^D + \eta_j^D \sum_{i \in \tilde{J} \setminus D} r_{ji}^D = \lambda r_{0j}^D + \sum_{i \in \tilde{J} \setminus D} \eta_i^D r_{ij}^D,$$

which is the traffic equation when nodes in  $D$  are down in  $\mathbf{Z}$  and has the unique solution  $(\eta_j^D, j \in \tilde{J} \setminus D)$ . The last step is: First observe that  $(\eta_j^D, j \in \tilde{J} \setminus D)$  is by Assumption 2.2 the restriction of  $(\eta_j, j \in \tilde{J})$  (the solution of the traffic equation with all nodes in  $\mathbf{Z}$  up), to  $\tilde{J} \setminus D$ , and, secondly, consider the above constructed system of independent birth-death processes for the reliability level  $\emptyset$  as the comparison system when all nodes are up with routing  $R'$ , and the  $R'^D$  as rerouting scheme for this network on reliability level  $D$ .

Because  $\eta_j = \eta_j'$  and  $\eta_j^D = \eta_j'^D$  for all  $j \in \tilde{J} \setminus D$  and all  $D$ , the  $\eta_j'^D$  are obtained by restricting  $(\eta_j' : j \in \tilde{J})$  (the solution of the traffic equation with all nodes in  $\mathbf{Z}'$  up), to  $\tilde{J} \setminus D$ .

Note that for all  $D$  holds  $R'^D \prec_P R^D$ , therefore  $R^D \prec_{pd} R'^D$ , and the result follows from Proposition 4.3.  $\square$

**Remark 4.6** *The overall balance (4.1) of the rerouting  $R^D$  for all network nodes which are up is satisfied whenever  $R$  is a reversible transition matrix.*

*If the overall balance (4.1) holds for  $\tilde{J}$  only (i.e., for  $D = \emptyset$ ), and the rerouting for  $\mathbf{Z}'$  is by stalling, the extremal property of  $\text{Id}$  for  $\prec_P$  leads to the same conclusion as the assumptions of Proposition 4.5.*



Remark: From irreducibility of  $\mathbf{Z}$ , for  $i, j \in \tilde{J}, i \neq j$  we obtain from (4.1)  $\eta_j r_{ji}^{\tilde{J} \setminus \{i,j\}} = \eta_i r_{ij}^{\tilde{J} \setminus \{i,j\}}$ , but this does not mean that the matrix  $R$  is reversible, because  $r_{ij}^{\tilde{J} \setminus \{i,j\}} \neq r_{ji}$  may hold.

The lower bound  $\text{Gap}(\mathbf{Z}')$  in the previous statements is of interest, because it has constitutive processes with conditionally independent coordinates. From [Lig89][Theorem 2.6] it is known, that the gap of a process with independent coordinates is the minimum of the gaps of the coordinate processes. Unfortunately enough, this theorem does not apply here directly, because the coordinate birth-death processes are controlled by the common breakdown-repair process. However, the comparison result of Proposition 4.3 can be used to obtain upper bounds for spectral gaps. This topic will be considered in a separate paper. Nevertheless the bound is of practical value, because the bounding process  $\mathbf{Z}'$  is reversible with respect to  $\tilde{\pi}$ , which can be seen by checking the local balance equations. As a consequence, the bounding techniques for reversible processes, e.g., using Cheeger constants, found in the literature can be applied directly.

In [LS14][Example 6.2] it is shown, that the bounds obtained via Proposition 4.5 can be very good for networks with reliable nodes. They compare the bound for an example provided in [IRT12]:

This is the network described in Section 2.1 with state independent service rates  $\mu_j = \mu$ , and routing matrix which fulfills  $r_{ii} = 0, \forall i = 0, 1, \dots, J$ , and  $r_{0i} > 0, \forall i = 1, \dots, J$ . Furthermore, for all  $i, j = 1, \dots, J, i \neq j$ , holds complete symmetry by  $r_{ij} = p \in (0, 1)/(j - 1)$ , which results in  $r_{i0} = 1 - p(J - 1) > 0, \forall i = 1, \dots, J$ .

It is assumed that no breakdowns (and repair) occur.

In this symmetric network the partial balance (for  $D = \emptyset$ ) holds if and only if  $r_{0i} = 1/J, \forall i = 1, \dots, J$ , which implies  $\lambda_j = \lambda/J, \forall j = 1, \dots, J$  and  $\eta_j = \lambda/(J(1 - p(J - 1))), \forall j = 1, \dots, J$ . We denote by  $\mu_{i_0} := \min\{\mu_j : 1 \leq j \leq J\}$ , and, recalling the bound of the spectral gap obtained for birth-death processes by van Doorn [Doo02], we obtain from the companion result of Proposition 4.5 (see [DS08][Proposition 4.4])

$$\text{Gap}(\mathbf{Z}') \geq \left( \sqrt{\mu_{i_0}(1 - p(J - 1))} - \sqrt{\frac{\lambda}{J}} \right)^2.$$

It is easy to check, that for this setting the Assumptions (3.10) and (3.11) of Corollary 3.4 in [IRT12] are fulfilled, which results in an upper bound for  $L^2$  spectral gap

$$\text{Gap}(\mathbf{Z}) \leq \frac{1 + p}{1 - p(J - 2)} \left( \sqrt{\mu_{i_0}(1 - p(J - 1))} - \sqrt{\frac{\lambda}{J}} \right)^2.$$

$\frac{1+p}{1-p(J-2)}$  tends monotonously to  $J$  for  $p \rightarrow 1/(J-1)$ , while for  $p \rightarrow 0$  it decreases monotonously to 1.

From the ordering implication *Peskun yields positive definiteness* follows, that if we perturb routing of customers in the networks by shifting transition probability mass from non diagonal entries into the diagonal (leaving the routing equilibrium fixed) then the speed of convergence of the perturbed process is smaller.

The existence of  $L^2$  spectral gap (that is the question when  $\text{Gap}(Q^{\mathbf{Z}}) > 0$ ) for unreliable networks is a related topic. It is a common knowledge that for networks with constant service rates (not depending on the number of customers at node) the spectral gap for classical

Jackson network exists. For service rates that can depend on the number of customers the problem is more delicate. An *iff* characterization in terms of properties of service rates is given in Lorek and Szekli [LS14]. A special feature of such processes is that the existence of  $L^2$  spectral gap is directly related to the tail properties of the stationary distribution. For references and details see [LS14].

An analogue of Peskun ordering and positive semidefinite order for generator matrices is as follows.

**Definition 4.7** Let  $Q = (q(x, y) : x, y \in \mathbb{E})$  and  $Q' = (q'(x, y) : x, y \in \mathbb{E})$  be generator matrices on a finite set  $\mathbb{E}$  such that  $\hat{\pi}Q = \hat{\pi}Q' = 0$  holds for a probability vector  $\hat{\pi}$ .

$Q'$  is smaller than  $Q$  in the positive semidefinite order for generators,  $Q' \prec_{pd} Q$ , if  $Q - Q'$  is positive semidefinite.

$Q'$  is smaller than  $Q$  in the Peskun order for generators,  $Q' \prec_P Q$ , if for all  $x, y \in \mathbb{E}$  with  $x \neq y$  holds  $q'(x, y) \leq q(x, y)$ .

**Lemma 4.8** Let  $Q = (q(x, y) : x, y \in \mathbb{E})$  and  $Q' = (q'(x, y) : x, y \in \mathbb{E})$  be generator matrices on a finite set  $\mathbb{E}$  such that  $\hat{\pi}Q = \hat{\pi}Q' = 0$  holds for a probability vector  $\hat{\pi}$ . Then  $Q \prec_P Q' \implies Q' \prec_{pd} Q$  holds. *Proof.* From  $q(x, y) \leq q'(x, y) \forall x, y \in \mathbb{E}$  with  $x \neq y$  follows for all  $x \in \mathbb{E}$  that  $q'(x, x) \leq q(x, x)$  holds. So  $Q' - Q := (q'(x, y) - q(x, y) : x, y \in \mathbb{E})$  is a generator matrix as well. Therefore  $-(Q' - Q)$  is positive semidefinite.  $\square$

A direct consequence of Definition 4.7, this lemma, and of Theorem 3.4 follows from (3.3).

**Corollary 4.9** Suppose  $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$  and  $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$  are ergodic unreliable Jackson network processes, having the same arrival and service intensities, and with the same routing regimes, described by  $R = [r_{ij}]_{i,j=0,1,\dots,J}$  and rerouting fulfilling Assumption 2.2.

The breakdown-repair process for  $\mathbf{Z}$  is given in Definition 2.1 and for  $\mathbf{Z}'$  is defined similarly via functions  $A', B' : 2^{\tilde{J}} \rightarrow [0, \infty)$ , as given in Theorem 3.4.

The processes  $\mathbf{Y}$  and  $\mathbf{Y}'$  are Markov with generators  $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K, L) : K, L \subseteq \tilde{J})$  of  $\mathbf{Y}$  and  $Q^{\mathbf{Y}'} = (q^{\mathbf{Y}'}(K, L) : K, L \subseteq \tilde{J})$  of  $\mathbf{Y}'$  as defined in (2.5) for  $\mathbf{Y}$  and similar for  $\mathbf{Y}'$ .

Assume that the stationary distributions of  $\mathbf{Y}$  and  $\mathbf{Y}'$  are identical, denoted by  $\hat{\pi}$ .

If  $Q^{\mathbf{Y}} \prec_{pd} Q^{\mathbf{Y}'}$  holds, then  $\text{Gap}(Q^{\mathbf{Y}'}) \leq \text{Gap}(Q^{\mathbf{Y}})$ .

An easy to understand property is that whenever the breakdown-repair process  $\mathbf{Y}$  of  $\mathbf{Z}$  is uniformly faster than the breakdown-repair process  $\mathbf{Y}'$  of  $\mathbf{Z}'$ , i.e., for all  $x \neq y$  holds  $q^{\mathbf{Y}'}(x, y) \leq q^{\mathbf{Y}}(x, y)$ , we have  $\text{Gap}(Q^{\mathbf{Y}'}) \leq \text{Gap}(Q^{\mathbf{Y}})$ . This follows directly from Lemma 4.8. So, for example, if we have  $A(D) = \kappa^{|D|} \cdot A'(D)$ , and  $B(D) = \kappa^{|D|} \cdot B'(D)$ ,  $A \in 2^{\tilde{J}}$ , for some  $\kappa > 1$ , then  $Q^{\mathbf{Y}'} \prec_P Q^{\mathbf{Y}}$ , and these breakdown-repair processes fulfill the requirement of Corollary 4.9.

## 4.2 Asymptotic variance

Peskun [Pes73] and Tierney [Tie98] derived comparison theorems for the asymptotic variance of Markov chains for application to optimal selection of MCMC transition kernels in discrete

time. These asymptotic variances occur as variance in the limiting distribution of central limit theorems (CLTs) for the MCMC estimators. For our network processes  $\mathbf{Z}$  we consider Markov chain  $(X_k, k \geq 1)$ , say with transition matrix  $K = Id + \varepsilon Q^{\mathbf{Z}}$  (with  $\varepsilon > 0$  sufficiently small). Under some regularity conditions on a homogeneous Markov chain with one step transition kernel  $K$  we can obtain CLT of the form

$$\sqrt{n} \left( \frac{1}{n} \sum_{k=1}^n f(X_k) - E_{\tilde{\pi}}(f(X_t)) \right) \xrightarrow{D} N(0, v(f, K)),$$

where the asymptotic variance is  $v(f, K) = \langle f, f \rangle_{\tilde{\pi}} - \tilde{\pi}(f) + 2 \sum_{k=1}^{\infty} \langle f, K^k f \rangle_{\tilde{\pi}}$ .

Regularity conditions under which CLT holds for such Markov chains is a topic which we shall study in a separate paper. For reversible chains with positive spectral gap it is possible to give conditions in terms of the service rates, but a general *iff* characterization in terms of the service rates seems to be an open problem.

**Proposition 4.10** *Consider two ergodic unreliable Jackson networks with the same arrival and service intensities, and state processes  $\mathbf{Z}$  and  $\mathbf{Z}'$ . Assume that the routing matrices  $R$  and  $R'$  are reversible with respect to  $\xi$ . Both networks follow a rerouting mechanism according to (2.8) with the property (2.9), such that  $R^D$  and  $R'^D$  are reversible with respect to  $\xi^D$ . If  $R^D$  and  $R'^D$  are ordered for all  $D$  in positive definite order,  $R'^D \prec_{pd} R^D$ , then for any function  $f \in L_0^2(\tilde{\mathbb{E}}, \tilde{\pi}) := \{g \in L^2(\tilde{\mathbb{E}}, \tilde{\pi}) : \pi(g) = 0\}$  holds  $v(f, Id + \varepsilon Q^{\mathbf{Z}}) \geq v(f, Id + \varepsilon Q^{\mathbf{Z}'})$ .*

Proof. For standard Jackson networks without breakdown and repair it is well known that reversibility of the routing matrix  $R$  implies reversibility of the joint queue length process. A direct way to prove this is to check the local balance equations with respect to the stationary distribution  $\pi$ . It is easy to see that this way of proof verifies reversibility of the processes  $\mathbf{Z}$  and  $\mathbf{Z}'$  here as well. The reason is that the breakdown and repair process is reversible with respect to  $\hat{\pi}$ , and that for fixed  $D$  and  $\mathbb{N}_D$  intensities of possible transitions on  $\mathbb{N}^{|\tilde{J} \setminus D|}$  balance locally with respect to the densities  $\prod_{\ell \in \tilde{J} \setminus D} \prod_{i=1}^{n_{\ell}} \left( \frac{\eta_{\ell}}{\mu_{\ell}(i)} \right)$ .

Because  $\mathbf{Z}$  and  $\mathbf{Z}'$  are irreducible we can apply a result of Mira and Geyer [MG99][Theorem 4.2], which states that under this condition the required inequality is equivalent to ordering of the one step correlations for  $f \in L_0^2(\mathbb{E}, \tilde{\pi})$ . The latter statement can be shown exactly as in the proof of Proposition 4.3.  $\square$

## 5 Jackson networks with finite waiting rooms

We consider the Jackson networks from Section 2.1 and assume now that some or all of the single server nodes with queue length dependent service intensities have a finite capacity. This means that such nodes have a service place and only a finite number of waiting places available to store waiting customers. The problem one is faced with is that whenever a node, say  $j$ , reaches its maximal buffer size then no further arriving customer can enter node  $j$  and we have to "resolve" such blocking situation.

There are many "blocking protocols" available in practice and in theoretical models but the standard observation is that when reducing the infinite buffers of the Jackson networks from

Section 2.1 to finite sizes the stationary distribution is no longer of product form, in fact in almost all cases the stationary distribution is not available. The simplest reaction to a full buffer situation with a newly arriving customer is to discard this customer - this leads to a simple equilibrium only in case of a single station network.

A survey, emphasizing applications in telecommunication systems is given in [Per90][Section 2], a general overview is [BDO01].

It turned out that for obtaining product form stationary distributions for finite buffer Jackson networks one has to introduce clever rerouting strategies, i.e. to redirect and redistribute newly arriving customers who want to enter nodes with full buffers in such a way that the the system's customer flow is balanced. A literature review shows that the most prominent rerouting strategies in case of full buffers which lead to product form equilibrium are the schemes mentioned in Example 2.5. These are described in more detail e.g. in [Dij11][Section 1.6.3] under the headings CONSERVATIVE PROTOCOL  $\equiv$  Stalling, JUMP-OVER PROTOCOL  $\equiv$  Skipping, and in [Per90][p. 456,457] as REPETITIVE-SERVICE-RANDOM DESTINATION. It turns out that in all three cases the stationary distribution for feasible states in the finite buffer network are (up to normalization) of the form (2.4). A detailed study of the jump-over protocol in case of blocking is [Dij88].

As a referee pointed out, it is therefore a tempting conjecture that correlation formulas similar to ours should be valid in the context of finite buffer networks as well. But going into the details shows that there are subtleties in the structure and the functioning of the finite buffer networks which must be handled careful.

(1) The network of Section 2.4 is influenced by an external environment which is Markov for its own, and changes of the environment enforces the Jackson network to redirect its customers, if necessary. In case of the finite buffer network the customers are redirected by intrinsic forces (state space restrictions).

(2) The intrinsic forces which influence routing and rerouting lead to queue length dependent routing decisions. Seemingly, this makes it nearly impossible to formulate an overall simple condition in parallel to Assumption 2.2, which produces product form stationary distributions.

(3) The area of applications (problems) which is the classical source of interest in finite buffer networks with blocking protocols is in telecommunications networks. There it is unrealistic to assume that in case of blocking (=full buffer) the service at the full buffer node is interrupted, which is natural at broken down nodes in the setting of Section 2.4.

Nevertheless, it turned out that correlation formulas similar to those in Propositions 3.2 and 3.1 can be derived in case of the most popular unblocking scheme (repetitive-service-random destination) for networks with reversible routing. This and more general results are part of our ongoing research and will be summarized in a forthcoming technical report [DS15].

## 6 Proof of the correlation formulas

For  $f, g : \mathbb{E} \rightarrow \mathbb{R}$  and steady state probability  $\tilde{\pi}$  of  $\mathbf{Z}$  we are interested in the one-step correlation expressions

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} &= C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^J \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, \underline{n}) \left\{ \sum_{j \in \tilde{J} \setminus D} \lambda_j^D g(D, \underline{n} + e_j) \right. \\ &+ \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) r_{j0}^D g(D, \underline{n} - e_j) + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D, i \neq j} \mu_j(n_j) r_{ji}^D g(D, \underline{n} - e_j + e_i) \\ &- \left( \sum_{j \in \tilde{J} \setminus D} \lambda_j^D + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) (1 - r_{jj}^D) \right) g(D, \underline{n}) \\ &\left. + \sum_{H \subset D} q^{\mathbf{Y}}(D, H) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) g(I, \underline{n}) - \left( \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right) g(D, \underline{n}) \right\}. \end{aligned} \quad (6.1)$$

Proof. (of Proposition 3.1) Interchanging summations, regrouping terms, and exploiting the product form structure of the state distribution, we obtain from (6.1)

$$\begin{aligned} &\sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \left[ \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \right. \\ &\quad \left. - f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right\} \right] \\ &+ \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \sum_{\underline{n}_D \in \mathbb{N}^{|D|}} \prod_{\ell \in D} \left( C_\ell^{-1} \prod_{i=1}^{n_\ell} \frac{\eta_\ell}{\mu_\ell(i)} \right) \\ &\left[ \sum_{\underline{n}_{\tilde{J} \setminus D} \in \mathbb{N}^{J-|D|}} \prod_{\ell \in \tilde{J} \setminus D} \left( C_\ell^{-1} \prod_{i=1}^{n_\ell} \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D})) \left\{ \sum_{j \in \tilde{J} \setminus D} \lambda_j^D g(D, \underline{n} + e_j) \right. \right. \\ &+ \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) r_{j0}^D g(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D} - e_j)) + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D, i \neq j} \mu_j(n_j) r_{ji}^D g(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D} - e_j + e_i)) \\ &\quad \left. \left. - \left( \sum_{j \in \tilde{J} \setminus D} \lambda_j^D + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) (1 - r_{jj}^D) \right) g(D, (\underline{n}_D, \underline{n}_{\tilde{J} \setminus D})) \right\} \right] \end{aligned}$$

For each fixed  $D, \underline{n}_D$ , the terms in the last squared brackets are identical to the one step correlation of a Jackson network in equilibrium on node set  $\tilde{J} \setminus D$  (with the respective transition rates) with respect to the functions  $f(D, (\underline{n}_D, (\cdot)_{\tilde{J} \setminus D}))$  and  $g(D, (\underline{n}_D, (\cdot)_{\tilde{J} \setminus D}))$ .

We have agreed to denote the generator of such network by  $Q^{\mathbf{X}_{\tilde{J} \setminus D}}$ , and its steady state by  $\pi_{\tilde{J} \setminus D}$ , which leads to the proposed formula with the aid of the synthetic networks.  $\square$

Proof. (of Proposition 3.2) We restart with the expression (6.1) and observe that for fixed  $D \subseteq \{1, \dots, J\}$  the contribution of  $-r_{jj}^D$  in the negative terms would be exactly the contribution in the double sum of  $i \in \tilde{J} \setminus D, i = j$  in the positive terms, where for  $i = j$  would occur  $g(D, \underline{n} - e_j + e_j) = g(D, \underline{n})$  otherwise. Together with  $\mu_j(0) = 0 \forall j$ , incorporating these contributions simplifies our expression to

$$\begin{aligned}
& -C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^J \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, \underline{n}) g(D, \underline{n}) \\
& \left[ \left( \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right) + \left( \sum_{j \in \tilde{J} \setminus D} \lambda_j^D + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) \right) \right] \\
& + C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^J \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) g(I, \underline{n}) \right\} \\
& + C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^J \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, \underline{n}) \left\{ \sum_{j \in \tilde{J} \setminus D} \lambda_j^D g(D, \underline{n} + e_j) \right. \\
& \left. + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) r_{j0}^D g(D, \underline{n} - e_j) + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D} \mu_j(n_j) r_{ji}^D g(D, \underline{n} - e_j + e_i) \right\}
\end{aligned} \tag{6.2}$$

The last two lines of the formula above turn into

$$\begin{aligned}
& C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \sum_{\underline{n}_D \in \mathbb{N}^D} \prod_{\ell \in D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) \\
& \left[ \sum_{\underline{n}_{\tilde{J} \setminus D} \in \mathbb{N}^{\tilde{J} \setminus D}} \prod_{\ell \in \tilde{J} \setminus D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D, \underline{n}) \left\{ \sum_{j \in \tilde{J} \setminus D} \lambda_j^D g(D, \underline{n} + e_j) \right. \right. \\
& \left. \left. + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) r_{j0}^D g(D, \underline{n} - e_j) + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D} \mu_j(n_j) r_{ji}^D g(D, \underline{n} - e_j + e_i) \right\} \right] \\
& = C^{-1} \sum_{D \subseteq \{1, \dots, J\}} \hat{\pi}(D) \sum_{\underline{n}_D \in \mathbb{N}^D} \prod_{\ell \in D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) \\
& \left[ \sum_{\underline{n}_{\tilde{J} \setminus D} \in \mathbb{N}^{\tilde{J} \setminus D}} \prod_{\ell \in \tilde{J} \setminus D} \prod_{i=1}^{n_\ell} \left( \frac{\eta_\ell}{\mu_\ell(i)} \right) \left\{ \sum_{j \in \tilde{J} \setminus D} f(D, \underline{n}) \lambda_j g(D, \underline{n} + e_j) r_{0j}^D \right. \right. \\
& \left. \left. + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) f(D, \underline{n}) g(D, \underline{n} - e_j) r_{j0}^D + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D} \mu_j(n_j) f(D, \underline{n}) g(D, \underline{n} - e_j + e_i) r_{ji}^D \right\} \right]
\end{aligned}$$

In the last line, if  $n_j > 0$ , the expression  $\mu_j(n_j)$  cancels against a factor in the steady state probability. This leads to (we underbrace some intuition and use  $\eta_j = \eta_j^D$  for  $j \in \tilde{J} \setminus D$ )

$$\begin{aligned} & \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \left\{ \sum_{j \in \tilde{J} \setminus D} \underbrace{\lambda f(D, \underline{n}) g(D, \underline{n} + e_j) r_{0j}^D}_{0 \rightarrow j} + \sum_{j \in \tilde{J} \setminus D} \underbrace{\eta_j^D f(D, \underline{n} + e_j) g(D, \underline{n})}_{j \rightarrow 0} \right\} \\ & + \sum_{j \in \tilde{J} \setminus D} \sum_{i \in \tilde{J} \setminus D} \underbrace{\eta_j^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) r_{ji}^D}_{j \rightarrow i} + \underbrace{\lambda f(D, \underline{n}) g(D, \underline{n}) r_{00}^D}_{0 \rightarrow 0} - \lambda f(D, \underline{n}) g(D, \underline{n}) r_{00}^D \end{aligned}$$

From Lemma 2.3, with  $\eta_0^D := \lambda$  the vector  $\hat{\eta}^D := (\eta_j^D, j \in \tilde{J}_0 \setminus D)$  solves  $x = x \cdot R^D$ . Insert the normalized solution  $\xi^D := (\xi_i^D : i \in \tilde{J}_0 \setminus D)$  into (6.3), and then insert into the correlation expressions (6.2) to obtain

$$\begin{aligned} \langle f, Q^Z g \rangle_{\tilde{\pi}} &= \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \left\{ \sum_{H \subset D} q^Y(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^Y(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \\ &+ \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j \in (\tilde{J} \cup \{0\}) \setminus D} \sum_{i \in (\tilde{J} \cup \{0\}) \setminus D} \xi_j^D r_{ji}^D f(D, \underline{n} + e_j) g(D, \underline{n} + e_i) \right\} \\ &- \sum_{D \subseteq \{1, \dots, J\}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) \lambda f(D, \underline{n}) g(D, \underline{n}) r_{00}^D \\ &- \sum_{D \subseteq \tilde{J}} \sum_{\underline{n} \in \mathbb{N}^J} \tilde{\pi}(D, \underline{n}) f(D, \underline{n}) g(D, \underline{n}) \left\{ \left[ \sum_{H \subset D} q^Y(D, H) + \sum_{I \supset D} q^Y(D, I) \right] + \left[ \sum_{j \in \tilde{J} \setminus D} \lambda_j^D + \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j) \right] \right\} \end{aligned}$$

With  $\sum_{j \in \tilde{J} \setminus D} \lambda_j^D + \lambda r_{00}^D = \lambda$  this yields finally the desired correlation formula from Proposition 3.2.  $\square$

**Acknowledgement.** We thank an anonymous referee for careful reading of the first version of the paper and her or his constructive critics.

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